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Viscoelastic flow past a wedge with a soluble coating

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The steady, two dimensional, incompressible flow of a viscoelastic fluid past a wedge of 90 degrees coated with the viscoelastic material is studied theoretically using constitutive equations proposed by Oldroyd in 1958. The effect of diffusion of the coating as well as its material properties (viscosity, relaxation time, retardation time, etc.) on the frictional force is investigated.

A boundary-layer analysis is performed on the constitutive equations as well as on the momentum equations. A similarity transformation is found for the set of boundary-layer equations. Series expansion and Laplace's method are employed to obtain the solution in the asymptotic series of gamma functions.

The results obtained show that:

(i) The thinner displacement thickness does not necessarily imply a large frictional force for the viscoelastic flow.

(ii) For a homogeneous viscoelastic flow, the frictional force increases as the degree of dilatancy of the material increases, and decreases with increasing degree of pseudo-plasticity of the material.

(iii) For a non-homogeneous viscoelastic flow with given material constants, depending on whether the material is pseudoplastic or dilatant and on the ratio of the material concentration of outer flow and the concentration at the body, the frictional coefficient will decrease or increase from that of the homogeneous flow with the concentration at the body as the Schmidt number increases, and will approach a limit when the Schmidt number becomes very large. This limit is the frictional coefficient of the homogeneous flow with the concentration of the outer flow.

1. Introduction

Many rheological models have been proposed to describe the mechanical behaviour of viscoelastic materials. In 1962 Williams & Bird discussed these proposed models and concluded that of the relatively simple ones, Oldroyd's model (1958) is the most reasonable one to represent viscoelastic liquids at the present time. They also used the model to study steady viscoelastic flow in tubes. By a proper choice of material constants they obtained results showing good agreement with experimental data up to moderate rates of shear (Williams & Bird 1962*a*; Fredrickson 1964). This model exhibits qualitatively the main non-Newtonian flow properties observed in flowing viscoelastic liquids such as polymers and colloidal solutions. Those properties are: a variable apparent viscosity which decreases with increasing rate of shear in simple shear, a Weissenberg climbing effect, and a Robert–Weissenberg normal stress pattern.

In this study Oldroyd's (1958) model has been used to describe the mechanical behaviour of viscoelastic materials for a steady, two-dimensional, incompressible flow past a semi-infinite flat plate coated with viscoelastic materials. The outer flow can be either a solvent or a solution of the coating. For the purpose of analysis it is assumed that the mixture of the coating in the main flow is of small enough concentration to have constant diffusivity as well as constant density. The governing equations of motion and diffusion are obtained by a boundary-layer analysis. It is found that the set of partial differential equations has a similarity solution only when the external stream velocity is proportional to the cube of the distance along the plate. This represents a flow of Falkner–Skan type past a wedge of 90 degrees.

2. Governing equations

A. Constitutive equations

For the idealized viscoelastic liquids considered here, the stress s_{ij} at any point in the flow may be considered as the superposition of two independent stress systems, that is

$$s_{ij} = -pg_{ij} + p_{ij},$$

in which g_{ij} are the components of the metric tensor, p is a scalar (not necessarily the pressure), and p_{ij} contains the non-isotropic part of the stress tensor. In 1958 Oldroyd proposed a mathematical model for p_{ij} which qualitatively describes many effects observed in real viscoelastic fluids. The proposed rheological equations of state relating p_{ij} and the rate of deformation tensor

$$d_{ij} \equiv \tfrac{1}{2}(\partial u_i/\partial x^j + \partial u_j/\partial x^i)$$

are

$$p_{ik} + \lambda_1(\mathscr{D}p_{ik}/\mathscr{D}t) - \mu_1(p_i^j d_{jk} + p_k^j d_{ij}) + \nu_1 p_{j\ell} d^{j\ell} g_{ik} + \mu_0 p_j^j d_{ik}$$
$$= 2\mu[d_{ik} + \lambda_2(\mathscr{D}d_{ik}/\mathscr{D}t) - 2\mu_2 d_{ij} d_k^j + \nu_2 d_{j\ell} d^{j\ell} g_{ik}]. \qquad (1)$$

Here μ, λ_1 and λ_2 are the viscosity, relaxation time, and retardation time of the material, respectively, at very small rates of strain, and μ_1, μ_2, ν_1, ν_2 and μ_0 are five material constants with the dimension of time. $\mathscr{D}/\mathscr{D}t$ is the Jaumann derivative, and u_i are the components of the velocity vector.

The Jaumann derivative is a time derivative of the components of a tensor as measured with respect to a rigid co-ordinate system which translates and rotates with a fluid particle. This derivative (as well as Oldroyd's convected derivative) satisfies the requirement of invariance of response in rheological equations of state. When the Jaumann derivative operates on a second-order tensor with components b_{ij} and is transformed to the fixed co-ordinates x^i, one has in Cartesian co-ordinates (e.g. Oldroyd 1958)

$$\frac{\mathscr{D}b_{ij}}{\mathscr{D}t} = \frac{\partial b_{ij}}{\partial t} + u^k \frac{\partial b_{ij}}{\partial x^k} - w_{ik} b_j^k + w_{kj} b_i^k$$

in which w_{ij} are the components of the vorticity tensor; i.e.

$$w_{ij} \equiv \tfrac{1}{2}[(\partial u_i/\partial x^j) - (\partial u_j/\partial x^i)].$$

The Jaumann derivative operating on either contravariant or covariant components of a tensor will result in the same form since

$$\mathscr{D}g_{ij}/\mathscr{D}t = 0 \quad \text{and} \quad \mathscr{D}g^{ij}/\mathscr{D}t = 0,$$

where g_{ij} are the components of the metric tensor. Thus, the use of the Jaumann derivative removes an objection against Oldroyd's convected derivative, which in general gives different forms for contravariant and covariant components of a tensor. In any case either the Jaumann or the Oldroyd derivative may be used in (1); the only difference is in interpretation of the parameters μ_1 and μ_2. Some of the implications of the constitutive equation (1) are given by Oldroyd (1958) and Williams & Bird (1962*a*).

In the present study a steady, two-dimensional, incompressible fluid flow problem is considered. Using Cartesian co-ordinates (x, y) directed along and perpendicular to the body, the constitutive equations (1) along with the continuity equation yield

$$\left.\begin{aligned}
&p_{xx}+\lambda_1\left[\frac{Dp_{xx}}{Dt}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)p_{xy}\right]-\mu_1\left[2\frac{\partial u}{\partial x}p_{xx}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)p_{xy}\right]\\
&\qquad+\nu_1\left[\frac{\partial u}{\partial x}p_{xx}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)p_{xy}+\frac{\partial v}{\partial y}p_{yy}\right]+\mu_0\frac{\partial u}{\partial x}(p_{xx}+p_{yy}+p_{zz})\\
&\quad=2\mu\left\{\frac{\partial u}{\partial x}+\lambda_2\left[\frac{D}{Dt}\left(\frac{\partial u}{\partial x}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right.\\
&\qquad\left.-2\mu_2\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{4}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]+\nu_2\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\right\},\\
&p_{xy}+\lambda_1\left[\frac{Dp_{xy}}{Dt}+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)(p_{yy}-p_{xx})\right]-\frac{\mu_1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(p_{xx}+p_{yy})\\
&\qquad\qquad+\frac{\mu_0}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)(p_{xx}+p_{yy}+p_{zz})\\
&\quad=2\mu\left\{\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\lambda_2\left[\frac{D}{Dt}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\right]\right\},\\
&p_{yy}+\lambda_1\left[\frac{Dp_{yy}}{Dt}+\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)p_{xy}\right]-\mu_1\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)p_{xy}+2\frac{\partial v}{\partial y}p_{yy}\right]\\
&\qquad+\nu_1\left[\frac{\partial u}{\partial x}p_{xx}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)p_{xy}+\frac{\partial v}{\partial y}p_{yy}\right]+\mu_0\left[\frac{\partial v}{\partial y}(p_{xx}+p_{yy}+p_{zz})\right]\\
&\quad=2\mu\left\{\frac{\partial v}{\partial y}+\lambda_2\left[\frac{D}{Dt}\left(\frac{\partial v}{\partial y}\right)+\frac{1}{2}\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]\right.\\
&\qquad\left.-2\mu_2\left[\left(\frac{\partial v}{\partial y}\right)^2+\frac{1}{4}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right]+\nu_2\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]\right\},\\
&p_{zz}+\lambda_1\frac{Dp_{zz}}{Dt}+\nu_1\left[\frac{\partial u}{\partial x}p_{xx}+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)p_{xy}+\frac{\partial v}{\partial y}p_{yy}\right]\\
&\quad=2\mu\nu_2\left[\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right],
\end{aligned}\right\} \tag{2}$$

where D/Dt is the substantial derivative, u and v are the velocity components in the x- and y-directions, respectively.

To make (2) dimensionless, the density ρ, the viscosity η_0, a characteristic free stream velocity U_∞, and a characteristic length L are chosen as reference quantities. A Reynolds number Re can then be defined as

$$Re = \rho U_\infty L/\eta_0 \equiv 1/\epsilon^2, \quad \text{say},$$

where $\epsilon \ll 1$ for the case under consideration. In order to perform a boundary-layer analysis, let

$$x = Lx', \quad y = \epsilon Ly', \quad u = U_\infty u', \quad v = \epsilon U_\infty v', \quad \mu = \eta_0 \mu', \tag{3}$$

$$(\lambda_1, \lambda_2, \mu_0, \mu_1, \mu_2, \nu_1, \nu_2) = \epsilon(L/U_\infty)(\lambda_1', \lambda_2', \mu_0', \mu_1', \mu_2', \nu_1', \nu_2'), \tag{4}$$

$$(p_{xx}, p_{xy}, p_{yy}, p_{zz}) = \epsilon\rho U_\infty^2(p_{xx}', p_{xy}', p_{yy}', p_{zz}'), \tag{5}$$

where primed quantities in (3) and (4) are assumed to be of order one or less. For the purposes of the present analysis, in writing (4) it was assumed that the order of the seven dimensionless material constants are all equal to or less than that of ϵ. This requirement ensures that the flow of a Newtonian fluid will be a limiting case. The order of magnitude of stresses shown in (5) is a consequence of (4) for large Reynolds number. This can be verified by the substitution of (3) and (4) into (2).

After the substitution of (3)–(5) into (2), one can, upon neglecting higher order terms in ϵ, obtain explicit forms for stresses in terms of the shear rate. They are, in dimensional forms:

$$p_{xx} = -\mu\left[(\lambda_2+\mu_2-\nu_2)-(\lambda_1+\mu_1-\nu_1)\frac{1+\Lambda_2(\partial u/\partial y)^2}{1+\Lambda_1(\partial u/\partial y)^2}\right]\left(\frac{\partial u}{\partial y}\right)^2, \tag{6}$$

$$p_{xy} = \mu\frac{\partial u}{\partial y}\frac{1+\Lambda_2(\partial u/\partial y)^2}{1+\Lambda_1(\partial u/\partial y)^2}, \tag{7}$$

$$p_{yy} = \mu\left[(\lambda_2-\mu_2+\nu_2)-(\lambda_1-\mu_1+\nu_1)\frac{1+\Lambda_2(\partial u/\partial y)^2}{1+\Lambda_1(\partial u/\partial y)^2}\right]\left(\frac{\partial u}{\partial y}\right)^2, \tag{8}$$

$$p_{zz} = \mu\left[\nu_2-\nu_1\frac{1+\Lambda_2(\partial u/\partial y)^2}{1+\Lambda_1(\partial u/\partial y)^2}\right]\left(\frac{\partial u}{\partial y}\right)^2, \tag{9}$$

in which

$$\Lambda_1 \equiv \lambda_1^2+\mu_0(\mu_1-\tfrac{3}{2}\nu_1)-\mu_1(\mu_1-\nu_1), \tag{10}$$

$$\Lambda_2 \equiv \lambda_1\lambda_2+\mu_0(\mu_2-\tfrac{3}{2}\nu_2)-\mu_1(\mu_2-\nu_2). \tag{11}$$

It is noted that (6)–(11) show that the state of stress at any point in the boundary layer is exactly the same as that in a steady simple shearing flow (Oldroyd 1958). Equation (7) indicates that in general the apparent viscosity, defined by

$$\mu_{\text{app.}} \equiv \frac{p_{xy}}{\partial u/\partial y} = \mu\left[1+\Lambda_2\left(\frac{\partial u}{\partial y}\right)^2\right]\Big/\left[1+\Lambda_1\left(\frac{\partial u}{\partial y}\right)^2\right] \tag{12}$$

depends on the shear rate and has the limiting values

$$\mu_{\text{app.}} \to \mu \quad \text{as} \quad \partial u/\partial y \to 0 \quad \text{and} \quad \mu_{\text{app.}} \to \mu(\Lambda_2/\Lambda_1) \quad \text{as} \quad \partial u/\partial y \to \infty.$$

Equations (6), (8) and (9) also show that the normal stresses are in general unequal, and that to obtain a two dimensional flow a stress p_{zz} normal to the flow has to be provided.

The seven material constants λ_1, μ_1, etc., and the viscosity μ are normally functions of the concentration of the viscoelectric component of the solution and hence can be represented by power series in the concentration c. Since the seven material constants will approach zero and the viscosity μ will approach a limiting value (η_0, say) when the concentration c approaches zero, it is reasonable to assume for a dilute material solution (i.e., for small values of c) that

$$\mu = \eta_0(1+\gamma c), \quad \lambda_1 = ac, \quad \mu_1 = bc, \quad \ldots, \quad \text{etc.} \tag{13}$$

Consequently (10) and (11) can be written as

$$\Lambda_1 = \beta c^2, \quad \Lambda_2 = \alpha c^2. \tag{14}$$

In (13) and (14), α, β, γ, etc. are constants for a given viscoelastic material. A more general representation of μ, λ_1, etc., in terms of c is possible within the scope of the similarity solution (see following), since any function of c alone is allowed by similarity for certain boundary distribution of c. However, the linearized form is felt sufficient to give an indication of the effects of the variation of the material parameters.

B. Similarity transformation

For flow rates at which (3)–(5) hold, one has a set of boundary layer equations

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{15}$$

$$u\,\partial u/\partial x + v\,\partial u/\partial y = U\,dU/dx + \rho^{-1}\partial p_{xy}/\partial y, \tag{16}$$

$$\partial(p - p_{yy})/\partial y = 0, \tag{17}$$

in which x and y are Cartesian co-ordinates along and perpendicular to the body, u and v are the components of mass average velocity in x- and y-directions and ρ is the mass average density of the solution. Since the problem considered is the one with diffusion process, the boundary-layer equation of diffusion is also one of the governing equations. In a similar manner it is found to be

$$u\,\partial \rho_c/\partial x + v\,\partial \rho_c/\partial y = \kappa\,\partial^2 \rho_c/\partial y^2, \tag{18}$$

where ρ_c is the mass density of the coating and κ is the mass diffusivity of the binary system.

The boundary conditions of the problem are

$$\text{at} \quad y = 0\colon \quad u = v = 0, \quad \rho_c = \rho_{c_0}; \tag{19}$$

$$\text{as} \quad y \to \infty\colon \quad u \to U(x), \quad \rho_c \to \rho_{c_1}. \tag{20}$$

It is assumed here that the dissolved viscoelastic material at the plate has a constant concentration ρ_{c_0}, and that the external flow is a solution of the coating with concentration ρ_{c_1}. When the outer fluid is Newtonian, ρ_{c_1} is zero.

The general solution of the system of partial differential equations (15)–(18) with the complicated shear stress-shear rate relation (7) is extremely difficult. A similarity transformation is sought to simplify the mathematics of the problem and to illustrate the general behaviour of the flow. From the continuity equation (15), a stream function $\psi(x, y)$ can be introduced such that

$$u = \partial\psi/\partial y, \quad v = -(\partial\psi/\partial x). \tag{21}$$

Letting

$$\eta = y\left(\frac{\rho}{\eta_0}\right)^{\frac{1}{2}}\frac{U(x)}{b(x)}, \quad \psi(x,y) = \left(\frac{\eta_0}{\rho}\right)^{\frac{1}{2}} b(x)f(\eta), \quad c \equiv \frac{\rho_c}{\rho} = a(x)g(\eta), \tag{22}$$

with the expression (7) for shear stress, the boundary-layer equations (16) and (18) for the dilute viscoelastic solution take the forms

$$UU_x f_\eta^2 - U^2\frac{b_x}{b}ff_{\eta\eta} - UU_x = \frac{U^3}{b^2}\frac{d}{d\eta}\left[(1+\gamma ag)f_{\eta\eta}\frac{1+\alpha(aU^2/b)^2(gf_{\eta\eta})^2}{1+\beta(aU^2/b)^2(gf_{\eta\eta})^2}\right], \tag{23}$$

$$a_x Uf_\eta g - aU\frac{b_x}{b}fg_\eta = \frac{1}{S}\frac{aU^2}{b^2}g_{\eta\eta}, \tag{24}$$

in which the subscripts x and η denote the differentiation with respect to x and η, respectively, and $S \equiv \eta_0/\kappa\rho$ is the Schmidt number. The velocity of the external flow, $U(x)$, and the two arbitrary functions $a(x)$ and $b(x)$ will be chosen such that (23) and (24) can be reduced to ordinary differential equations. This implies that the velocity profiles as well as the concentration profiles are similar at all the positions x.

The two possible similarity transformations of the problem are discussed as the following:

(i) If the velocity of the outer flow, U, varies with x, (23) and (24) can be written as:

$$f_\eta^2 - mff_{\eta\eta} - 1 = \kappa_2\frac{d}{d\eta}\left[(1+\gamma\kappa_1\kappa_3 U^{m-2}g)f_{\eta\eta}\frac{1+\alpha\kappa_3^2(gf_{\eta\eta})^2}{1+\beta\kappa_3^2(gf_{\eta\eta})^2}\right], \tag{25}$$

$$(m-2)f_\eta g - mfg_\eta = (\kappa_2/S)g_{\eta\eta}, \tag{26}$$

respectively, if one chooses

$$b(x) = \kappa_1 U^m(x), \quad a(x) = \kappa_1\kappa_3 U^{m-2}, \quad U = \left[\frac{(2m-1)x}{\kappa_1^2\kappa_2}\right]^{1/(2m-1)}, \tag{27}$$

where κ_i and m are arbitrary constants. Equation (25) indicates that if the material constant γ is zero or if m is equal to 2, the set of partial differential equations (15)–(18) can be transformed into a set of ordinary non-linear differential equations. In order to satisfy the similarity requirement the concentration at the boundary has to be proportional to $a(x) \equiv [U(x)]^{m-2}$. Thus for m equal to 2 the concentration at the body is a constant.

(ii) When the external flow has no pressure gradient and U is a constant, (23) and (24) become

$$-\kappa_1 ff_{\eta\eta} = \frac{d}{d\eta}\left[(1+\gamma\kappa_2 bg)f_{\eta\eta}\frac{1+\alpha U^4\kappa_2(gf_{\eta\eta})^2}{1+\beta U^4\kappa_2(gf_{\eta\eta})^2}\right], \tag{28}$$

$$f_\eta g - fg_\eta = \frac{1}{S\kappa_1}g_{\eta\eta}, \tag{29}$$

where now

$$b(x) = (2\kappa_1 Ux)^{\frac{1}{2}}, \quad a(x) = \kappa_2 b(x), \tag{30}$$

where κ_1 and κ_2 are arbitrary constants. Again, if γ is zero, the boundary-layer equation of motion and diffusion for flow past a semi-infinite flat plate with zero pressure gradient can be reduced to ordinary differential equations. The concentration at the body for this case now has to be proportional to $x^{\frac{1}{2}}$ in order to make the transformation valid for the problem.

It is clear that the two possible transformations are valid whenever the viscosity μ and the seven material constants λ_1, μ_1, etc., are all proportional to a power of the concentration.

C. Governing differential system

In general the viscosity of a viscoelastic liquid depends on the concentration of the solution. Therefore, if this is to be included in the problem, from the above analysis the only similarity transformation one can have for the problem is that when m equals 2. Now if the arbitrary constants κ_i are chosen such as

$$\kappa_1 = E^{-\frac{3}{2}}, \quad \kappa_2 = 3, \quad \kappa_3 = c_0/\kappa_1,$$

then (22) and (27) give

$$U(x) = Ex^{\frac{1}{3}}, \quad \eta = y\left(\frac{U\rho}{x\eta_0}\right)^{\frac{1}{2}}, \quad \psi(x,y) = \left(\frac{Ux\eta_0}{\rho}\right)^{\frac{1}{2}} f(\eta), \quad c = c_0 g(\eta). \tag{31}$$

This transformation implies that the flow problem is a special type of flow, that is a flow of Falkner–Skan type past a 90 degree wedge (Meskyn 1961). The transformed governing differential system for this special flow is

$$f_\eta^2 - 2ff_{\eta\eta} - 1 = 3\frac{d}{d\eta}[(1+Rg)f_{\eta\eta}(1+Ag^2f_{\eta\eta}^2)/(1+Bg^2f_{\eta\eta}^2)], \tag{32}$$

$$-\tfrac{2}{3}Sfg_\eta = g_{\eta\eta}. \tag{33}$$

where $R \equiv \gamma c_0$, $A \equiv \alpha c_0^2 E^3 \rho/\eta_0$, $B \equiv A\beta/\alpha$. The boundary conditions (19) and (20) now become

$$\text{at} \quad \eta = 0\colon \quad f = f_\eta = 0, \quad g = 1, \tag{34}$$

$$\text{as} \quad \eta \to \infty\colon \quad f_\eta \to 1, \quad g \to g(\infty) = c_1/c_0. \tag{35}$$

The coefficient of skin friction, C_d, will be given by

$$C_d \equiv (\rho/\eta_0 E^3)^{\frac{1}{2}} P_{xy}|_{\eta=0} = (1+R)f_{\eta\eta}(0)[1+Af_{\eta\eta}^2(0)]/[1+Bf_{\eta\eta}^2(0)]. \tag{36}$$

It is seen that the shear stress is a constant along the boundary for this special flow.

3. Method of solution for small values of the Schmidt number

The governing differential equations (32) and (33) with the boundary conditions (34) and (35) are next solved. Since the general solution of (32)–(35) cannot be obtained in terms of known functions, it is necessary to use either purely numerical methods or series expansions.

The method of series expansions accompanied by Laplace's method is employed here to solve the problem. This method was first used by Meksyn (1956) to solve the boundary-layer equation for a Newtonian fluid. Several classical problems for Newtonian fluids have been reworked by this method, and the results obtained are very striking in that only a few terms in the expansion are sufficient to obtain close agreement with accepted numerical results. Later Jones (1961) also used the method to study the boundary-layer equations of inelastic liquids flowing past a wedge.

In applying the method to solve (32) and (23), one first expresses the dependent variables $f(\eta)$ and $g(\eta)$ in power series of η such that

$$f(\eta) = \sum_{n=0}^{\infty} \frac{A_n}{n!}\eta^n, \quad g(\eta) = \sum_{n=0}^{\infty} \frac{B_n}{n!}\eta^n, \tag{37), (38}$$

in which A_n and B_n are constant coefficients to be determined. The expansions in (37) and (38) are valid only for sufficiently small values of η. By using the boundary conditions (34) one finds, from (37) and (38) that

$$A_0 = A_1 = 0, \quad B_0 = 1.$$

Substituting the expansions (37) and (38) into (33), the coefficients of the same power of η in both sides of the equation must be identical. This gives the relations between the B_n and A_n, which are found to be:

$$B_2 = 0, \quad B_3 = 0, \quad B_4 = -2SA_2B_1/3, \quad \text{etc.} \tag{39}$$

Similarly, the substitution of (37)–(39) into (32) yields

$$\left.\begin{aligned} A_3 &= [1+3B_1A_2M_0+BA_2^2-2B_1A_2^2I_0]/\text{Det},\\ A_4 &= 2[3B_1A_3M_0-3KBE_1^2-2A_2E_1I_1-(E_1^2+2A_2B_1A_3)I_0]/\text{Det, etc.}\end{aligned}\right\} \tag{40}$$

in which

$$\begin{aligned} \text{Det} &\equiv 2A_2I_0-3AA_2^2(1+R)+3(2KBA_2-1-R),\\ I_0 &\equiv -3AA_2(1+R), \quad K \equiv \frac{A_2(1+R)(1+AA_2^2)}{1+BA_2^2},\\ M_0 &\equiv R(1+AA_2^2)-2KBA_2,\\ I_1 &\equiv -B-3AA_3(1+R)-3ARB_1A_2, \quad E_1 \equiv A_3+A_2B_1.\end{aligned}$$

Equations (39) and (40) indicate that all of the expansion coefficients A_n and B_n can be related to A_2 and B_1 for given parameters A, B, R and S. The remaining unknown coefficients A_2 and B_1 are determined by the boundary conditions (35).

Integration of the diffusion equation (33) twice gives

$$g(\eta) = 1+B_1\int_0^\eta e^{-F(\eta)}d\eta \tag{41}$$

in which
$$F(\eta) = \frac{2S}{3}\int_0^\eta f(\eta)\,d\eta, \quad B_1 = [g(\infty)-1]\Big/\int_0^\infty e^{-F(\eta)}d\eta. \tag{42}$$

It is seen from the series expansion that the integral in (41) has a col (saddle point) at $\eta = 0$. Laplace's method (Copson 1946) can then be used to provide an asymptotic expansion for $g(\eta)$, that is

$$g(\eta) \sim 1+\tfrac{1}{3}B_1\sum_{m=0}^{\infty} b_m\Gamma_\tau\left(\frac{1+m}{3}\right), \tag{43}$$

in which Γ_τ are incomplete gamma functions; the corresponding value of η for a given τ can be obtained from the relation

$$\eta = \sum_{m=0}^{\infty}\frac{b_m}{1+m}\tau^{\frac{1}{3}(1+m)}. \tag{44}$$

The coefficients b_m in (43) and (44) are found to be

$$b_0 = \left(\frac{9}{SA_2}\right)^{\frac{1}{3}}, \quad b_1 = -b_0^2\frac{A_3}{6A_2}, \quad b_2 = b_0^3\left[-\frac{A_4}{20A_2}+\frac{1}{16}\left(\frac{A_3}{A_2}\right)^2\right], \quad \text{etc.} \tag{45}$$

Application of the boundary conditions (35) gives

$$B_1 \sim 3[g(\infty)-1] \Big/ \sum_{m=0}^{\infty} b_m \Gamma\left(\frac{1+m}{3}\right). \tag{46}$$

This is the first of the two relations which will be used to determine the unknown coefficients A_2 and B_1. To obtain the second relation the boundary-layer equation of motion (32) is used. This equation can be rewritten in the form

$$f_{\eta\eta\eta} + \frac{2f}{3(1+Rg)} f_{\eta\eta} = H(\eta), \tag{47}$$

in which

$$H(\eta) = \frac{1}{3(1+Rg)} \left\{ f_\eta^2 - 1 - 3Rg_\eta f_{\eta\eta} + \frac{d}{d\eta}\left[(1+Rg)f_{\eta\eta}\frac{(B-A)g^2 f_{\eta\eta}^2}{1+Bg^2 f_{\eta\eta}^2}\right]\right\}.$$

Letting $\quad F_1(\eta) = \frac{2}{3}\int_0^\eta \frac{f}{1+Rg}\,d\eta, \quad \phi(\eta) = f_{\eta\eta}(0) + \int_0^\eta e^{F_1(\eta)} H(\eta)\,d\eta,$

integration of (47) once yields

$$f_{\eta\eta}(\eta) = e^{-F_1(\eta)}\phi(\eta), \tag{48}$$

and a further integration gives

$$f_\eta(\eta) = \int_0^\eta e^{-F_1(\eta)}\phi(\eta)\,d\eta, \tag{49}$$

a form which can be evaluated by Laplace's method.

Before applying Laplace's method to evaluate (49), the properties of its integrand should be investigated. Since the form of $F(\eta)$ indicates that the integral in (42) has a col of order two at $\eta = 0$, the function $F_1(\eta)$ will dictate that the integral (49) also has a col of order two at $\eta = 0$. $F_1(\eta)$ is also known to be a positive function. When η becomes very large, the equation of motion takes the form

$$f_{\eta\eta\eta} + \frac{2\eta}{3[1+Rg(\infty)]} f_{\eta\eta} \simeq 0, \tag{50}$$

because $f_\eta \to 1$, $f \to \eta$, $g \to g(\infty)$ and $f_{\eta\eta} \to 0$ as $\eta \to \infty$. Integration of (50) yields

$$f_{\eta\eta} \sim \text{constant} \times \exp\{-\eta^{\frac{2}{3}}[1+Rg(\infty)]\}. \tag{51}$$

Thus, the comparison between (48) and (51) shows that $\phi(\eta)$ approaches a constant value as $\eta \to \infty$, and can be expected to be a slowly varying function of η throughout most of the region for small values of the Schmidt number. However, for a large Schmidt number the function $H(\eta)$ shows that $\phi(\eta)$ will be a rapidly varying function near the boundary because g_η changes very rapidly near the boundary. This implies that the integral in (49) cannot be approximated by Laplace's method when the Schmidt number is large unless the integrand is rearranged. It will be shown in the next section that the outer flow past the body without diffusion is the limit when the Schmidt number approaches infinity, and that an expansion of the inner and outer type is better suited for Schmidt numbers of the order of 10 or greater.

The integral in (49) can now be evaluated asymptotically by Laplace's method for small Schmidt numbers. It is found that

$$f_\eta(\eta) \sim \sum_{m=0}^{\infty} h_m \Gamma_\tau\left(\frac{1+m}{3}\right), \tag{52}$$

where the coefficients h_m are

$$\left.\begin{aligned} h_0 &= \tfrac{1}{3}A_2\left[\frac{A_2}{9(1+R)}\right]^{-\frac{1}{3}}, \quad h_1 = \left[\frac{A_3}{6}+\frac{A_2}{2}\left(\frac{RB_1}{1+R}\right)\right]\left[\frac{A_2}{9(1+R)}\right]^{-\frac{2}{3}}, \\ h_2 &= \left[-\frac{A_2}{80}\left(\frac{RB_1}{1+R}\right)^2+\frac{23A_3}{120}\left(\frac{RB_1}{1+R}\right)+\frac{3A_4}{20}-\frac{A_3^2}{16A_2}\right]\left[\frac{A_2}{9(1+R)}\right]^{-1}, \quad \text{etc.} \end{aligned}\right\} \tag{53}$$

The corresponding value of η for a given τ can be obtained from the relation

$$\eta = \sum_{m=0}^{\infty} \frac{b'_m}{1+m}\,\tau^{\frac{1}{3}(1+m)}, \tag{54}$$

with the coefficients

$$\left.\begin{aligned} b'_0 &= \left[\frac{A_2}{9(1+R)}\right]^{-\frac{1}{3}}, \quad b'_1 = (b'_0)^2\left[\frac{RB_1}{2(1+R)}-\frac{A_3}{6A_2}\right], \\ b'_2 &= (b'_0)^3\left\{-\frac{3}{5}\left(\frac{RB_1}{1+R}\right)^2+\frac{A_3}{5A_2}\left(\frac{RB_1}{1+R}\right)-\frac{A_4}{20A_2}+\left[\frac{3RB_1}{4(1+R)}-\frac{A_3}{4A_2}\right]^2\right\}, \quad \text{etc.} \end{aligned}\right\} \tag{55}$$

Applying boundary condition (35) to (52) yields

$$1 \sim \sum_{m=0}^{\infty} h_m \Gamma\left(\frac{1+m}{3}\right). \tag{56}$$

This relation together with (46) is used to determine the two unknown constants A_2 and B_1 for given values of the parameters A, B, R and S.

It is of course possible to determine A_2 and B_1 in an analytical form from the two relations (46) and (56) if only a finite number of terms in the relations are considered. However, since b_m in (46) and h_m in (56) are rather complicated functions of A_2 and B_1, the expression for general values of the parameters is much too cumbersome to obtain, and a trial and error procedure would be necessary to determine A_2 and B_1. For a large Schmidt number, (45) indicates that the series in the denominator of (46) will converge rather rapidly, but for a small Schmidt number the series in (46) is a divergent asymptotic series; therefore, only the first few converging terms should be considered in the determination of A_2 and B_1. After values of A_2 and B_1 have been determined, the concentration distribution of the coating and the velocity profile can be obtained from (43) and (52), respectively. The coefficient of the skin friction can then be calculated from (36), in which $f_{\eta\eta}(0) \equiv A_2$.

The series of (54), and (52) and (56), which involve the coefficients b'_m and h_m, are in general divergent. These divergences arise because the integral in (49) does not involve a large parameter and because the expansion used for $\phi(\eta)$ has a small radius of convergence about $\eta = 0$. The sum of a divergent series cannot be obtained directly. However, since the sum of a divergent series is the finite numerical value of the convergent expression from which the divergent series is derived (Euler 1755), it is possible, by a suitable transformation of the series, to obtain an asymptotic series which sums to the correct value. In the present study Euler's transformation is used. In the evaluation of a divergent series, Euler's transformation can be repeatedly applied until a convergent expression is obtained, and the transformation can be started at any term of the original or the transformed series. However, if only a finite number of terms is used in summing the series, too many repeated transformations will reduce the accuracy of the sum because the convergence of these first few terms will be slowed down by repeated transformation.

4. Results and discussion

Case I. *Viscoelastic liquids with homogeneous properties* ($g = 1$ *throughout the flow region*) *flowing past the wedge.*

For this case the coefficients B_n are all zero, since $g = 1$. To determine the correct value of A_2, eight terms of the series (56) are considered. It was necessary to transform this divergent series twice using Euler's transformation. The first

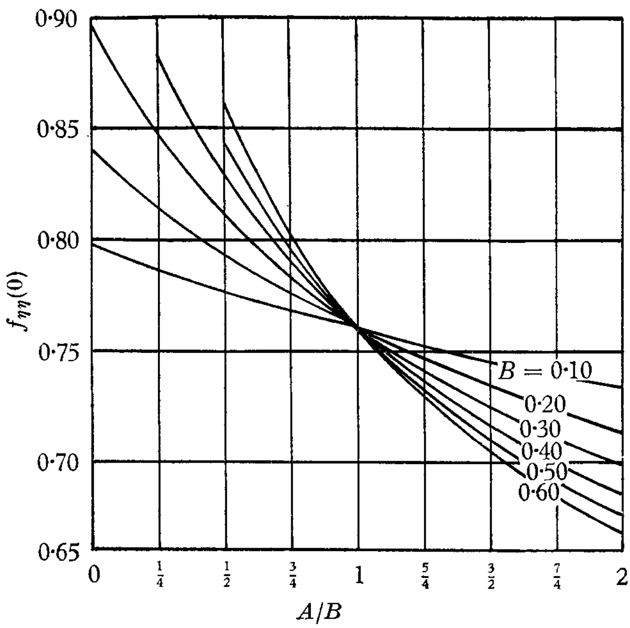

FIGURE 1. Effect of the parameters A and B on the slope of the velocity profile at the body when $R = 0$.

transformation was started at the very first term of the series, and the second transformation at the second term of the transformed series. It is noted from the boundary-layer equation of motion that when $R = 0$ and $A/B = 1$, the non-Newtonian phenomena will not be observed, that is the flow pattern of this case is the same as the one due to a Newtonian liquid. The results obtained for various values of A and B are shown in figures 1 and 2.

Figure 1 shows the relation between $f_{\eta\eta}(0)$ and the ratio A/B for $B = 0{\cdot}10$–$0{\cdot}60$ when the parameter R is zero. Since $f_{\eta\eta}(0)$ is the slope of the velocity profile at the body, it is legitimate to say that a larger value of $f_{\eta\eta}(0)$ implies a thinner displacement thickness. Thus, figure 1 shows that the displacement thickness increases with increasing A/B for a given B. For the case of a Newtonian liquid, that is when $A/B = 1$, the obtained result $f_{\eta\eta}(0) = 0{\cdot}761$ is in very good agreement with Hartree's (1937) result (0·758). This implies that the method used is probably quite accurate.

Figure 2 shows the relation between the coefficient C_d of the skin friction and the ratio A/B for $R = 0$. In the Newtonian flow problem, it is well known that the frictional coefficient C_d is linearly proportional to $f_{\eta\eta}(0)$ under the conditions of

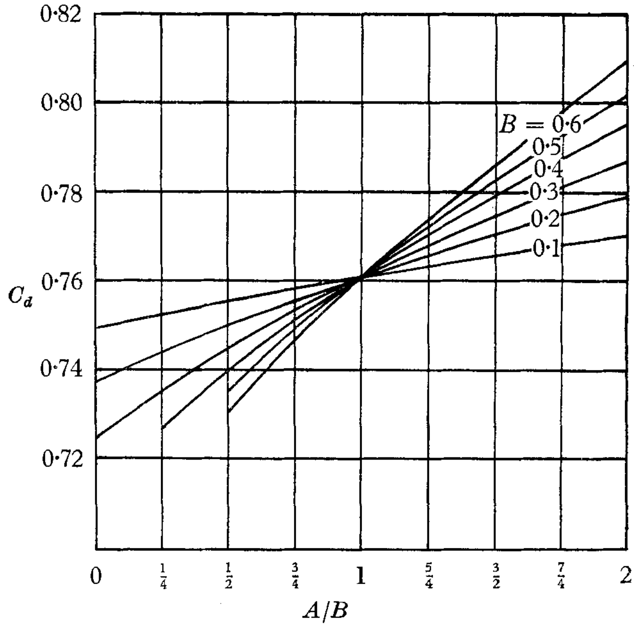

FIGURE 2. Effect of the parameters A and B on the frictional coefficient C_d when $R = 0$.

the boundary-layer assumption, but the results obtained here for viscoelastic liquids do not show such a simple relation. While $f_{\eta\eta}(0)$ decreases with increasing A/B as shown in figure 1, figure 2 indicates that C_d increases with increasing A/B. Therefore the skin friction of a viscoelastic liquid past the body is rather strongly affected by the values of the material constants λ_1, λ_2, μ_1, etc., and a displacement thickness thinner than that in a Newtonian fluid does not necessarily imply a larger frictional force for viscoelastic liquids.

The general expressions for normal stresses in terms of given A and B given by (6), (8) and (9) show a rather complex dependency on the seven material constants. For the special case which has been shown to predict the general form of some experimentally observed relations between steady state and oscillatory phenomena, suggested by Williams & Bird (1962*b*), that is when

$$\mu_1 = \lambda_1, \quad \nu_1 = \tfrac{2}{3}\lambda_1, \quad \mu_0 = 0, \quad \mu_2 = \lambda_2, \quad \nu_2 = \tfrac{2}{3}\lambda_2,$$

equations (6)–(9) become, after the similarity transformation,

$$\left.\begin{aligned} p_{xx} &= -2p_{yy} = -2p_{zz} = \frac{4\rho E^3}{3\eta_0}\mu f_{\eta\eta}^2(\lambda_1-\lambda_2)\Big/\left(1+\frac{2\rho E^3}{3\eta_0}\lambda_1^2 f_{\eta\eta}^2\right), \\ p_{xy} &= \left(\frac{\rho E^3}{\eta_0}\right)^{\frac{1}{2}}\mu f_{\eta\eta}\left(1+\frac{2\rho E^3}{3\eta_0}\lambda_1\lambda_2 f_{\eta\eta}^2\right)\Big/\left(1+\frac{2\rho E^3}{3\eta_0}\lambda_1^2 f_{\eta\eta}^2\right). \end{aligned}\right\} \qquad (57)$$

Thus, for pseudoplastic fluids ($A/B < 1$, i.e. apparent viscosity decreases with increasing rate of shear), p_{xx} is a tensile stress while p_{yy} and p_{zz} are compressive stresses; for dilatant fluids ($A/B > 1$, i.e. apparent viscosity increases with increasing rate of shear), p_{xx} becomes a compressive stress while p_{yy} and p_{zz} become tensile stresses. Hence, in order to have a steady, two dimensional, incompressible viscoelastic flow, depending upon whether the fluid is a pseudoplastic or a dilatant liquid, it is necessary to apply a compressive or a tensile stress normal to the plane of the flow.

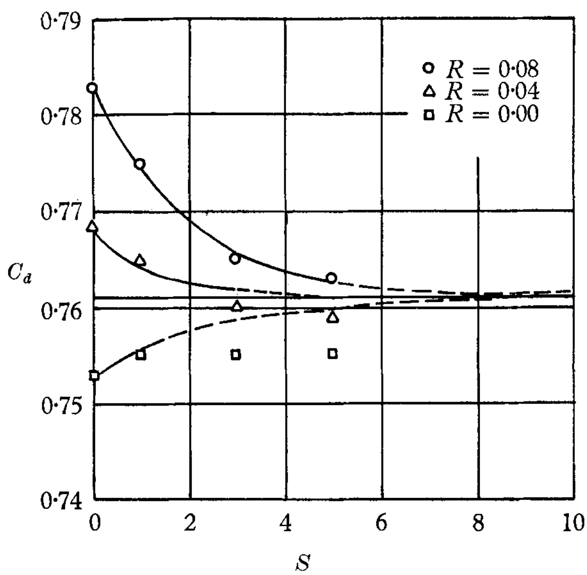

FIGURE 3. Relationship between the Schmidt number S and the frictional coefficient C_d for pseudoplastic coatings with $3B = 4A = 0·9$.

The effect of the parameter R on either $f_{\eta\eta}(0)$ or the frictional coefficient C_d can be obtained from the results given in figures 1 and 2 by a simple modification. It is seen that if the characteristic viscosity used in forming the dimensionless quantities is taken to be $\eta_0(1+R)$, the equation of motion then obtained is independent of R. Thus, for given values of the parameters A, B and R, the corresponding $f_{\eta\eta}(0)$ and C_d can be found, using figures 1 and 2, from the relations

$$\left.\begin{aligned} f_{\eta\eta}(\eta = 0, R) &= f_{\eta\eta}(\eta = 0, R = 0)/[1+R]^{\frac{1}{2}}, \\ C_d(R) &= [1+R]^{\frac{1}{2}} C_d(R = 0), \end{aligned}\right\} \tag{58}$$

respectively, for given values of A and B.

Case II. *Newtonian solvents flowing past the coated wedge*

Here the case is considered when the Schmidt number is larger than zero and $g(\infty)$ in (46) is equal to zero. Figures 3, 4 and 5 show the relationship between the frictional coefficient C_d and the Schmidt number S for $B = 0·3$, $R = 0·00–0·08$ with the ratio $A/B = \frac{3}{4}, 1, \frac{4}{3}$ respectively. The results obtained here indicate that the frictional coefficient C_d will increase or decrease from the corresponding value of the homogeneous viscoelastic flow with concentration c_0 and approach a limit when the Schmidt number increases. It will next be shown that this limit is the frictional coefficient of a Newtonian liquid past the wedge.

For a large Schmidt number, the diffusion layer is much thinner than the Prandtl boundary layer. Thus the boundary layer may be divided into two

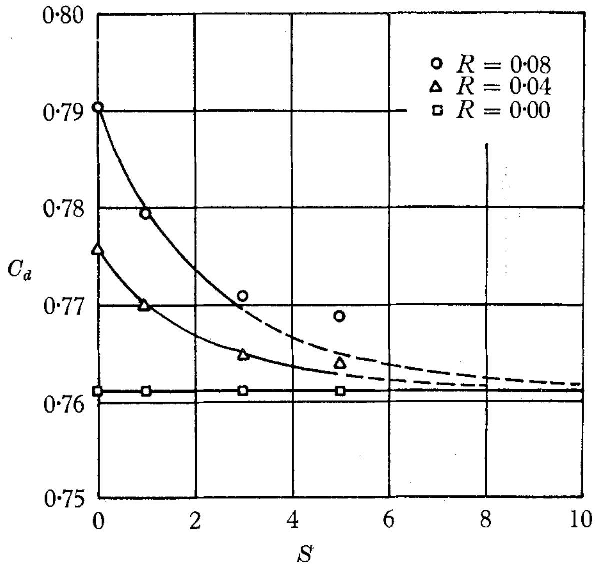

FIGURE 4. Relationship between the Schmidt number S and the frictional coefficient C_d when $A = B$.

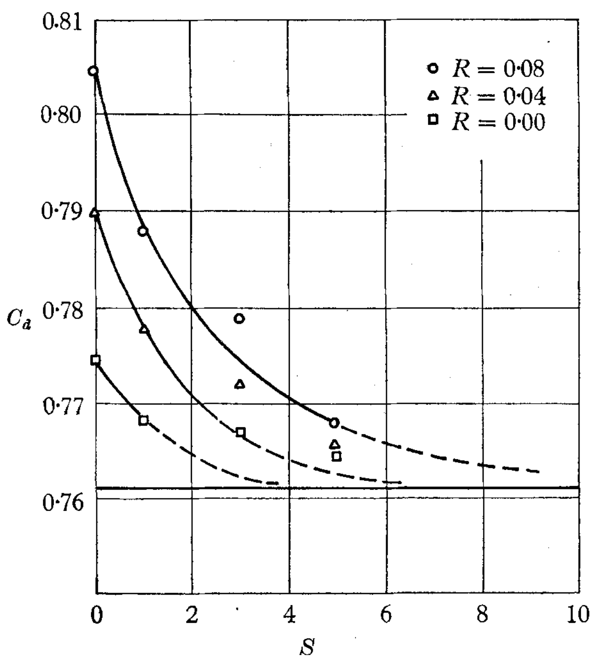

FIGURE 5. Relationship between the Schmidt number S and the frictional coefficient C_d for dilatant coatings with $3B = 2A = 0·9$.

regions, the first being a region of constant concentration far from the boundary, the second, a region of rapidly changing concentration in the immediate vicinity of the coated surface. For the case considered here the concentration of the first region is $g(\infty) = 0$, and the governing differential equation for the region is

$$f_\eta^2 - 2ff_{\eta\eta} - 1 = 3f_{\eta\eta\eta}.$$

The governing equations of the thin diffusion layer are (32) and (33). Since the diffusion term of (33) is comparable to the convective term and $f_{\eta\eta}(\eta)$ is expected to be order one in the diffusion layer, the transformation

$$\zeta = S^{\frac{1}{3}}\eta, \quad F = S^{\frac{2}{3}}f \tag{59}$$

should be chosen for a large Schmidt number. By the transformation (59), equations (32) and (33) become

$$d/d\zeta[F_{\zeta\zeta}(1+Rg)(1+Ag^2F_{\zeta\zeta}^2)/(1+Bg^2F_{\zeta\xi}^2)] = O(S^{-\frac{1}{3}}), \tag{60}$$

$$-\tfrac{2}{3}Fg_\zeta = g_{\zeta\zeta}. \tag{61}$$

Integrating (60) once, one has

$$F_{\zeta\zeta}(1+Rg)(1+Ag^2F_{\zeta\zeta}^2)/(1+Bg^2F_{\zeta\zeta}^2) = \text{const.} \equiv \tau_0 \quad \text{for} \quad S \to \infty. \tag{62}$$

This shows that the thin diffusion layer has a constant shear stress τ_0. From (61) it is known that g decays rapidly and approaches zero as $\zeta \to \infty$, thus (62) can be written as

$$F_{\zeta\zeta} = \tau_0 \quad \text{as} \quad \zeta \to \infty \quad \text{and} \quad S \to \infty. \tag{63}$$

Integration of (63) yields

$$\left.\begin{aligned} F_\zeta &= \tau_0\zeta + \text{const.}, \\ F &= \tfrac{1}{2}\tau_0\zeta^2 + \text{const.} \times \zeta + \text{const.}_1. \end{aligned}\right\} \tag{64}$$

To match the solutions of the two regions it is required that

$$f_{\eta\eta}(0) = F_{\zeta\zeta}(\infty).$$

This implies that τ_0 is the dimensionless shear stress at the body for Newtonian flow past the wedge. Furthermore, (64) satisfies the matching conditions

$$f(0) = f_\eta(0) = 0,$$

Hence as the Schmidt number approaches infinity, the frictional coefficient C_d has to approach that of the Newtonian case.

The above analysis shows that the frictional coefficient C_d will approach that of a Newtonian fluid as the Schmidt number becomes large. Hence based on the calculated results, the curves in figures, 3, 4 and 5 can be extended smoothly to approach the Newtonian limit. Due to the fact that only the finite numbers of terms are used to obtain the results, some of the calculated results shown in figures 3, 4 and 5 are away from the expected curves. However, the deviation in all cases is less than 1 % of the total C_d.

Figures 6, 7 and 8 show the relation between the frictional coefficient C_d and the ratio of parameters A and B for Schmidt numbers $S = 0$, 1, 3 and 5 when $B = 0{\cdot}3$ and $R = 0{\cdot}00$, $0{\cdot}04$ and $0{\cdot}08$. For $S = 0$ and 1 the calculated results are probably quite good, but when the Schmidt number becomes large, the deviation increases due to the method of solution. The curves shown in these figures have been adjusted according to figures 3, 4 and 5 for Schmidt numbers greater than 3.

From the results obtained in this case one can conclude that for dilatant coatings the frictional coefficient C_d will decrease with increasing Schmidt number and approach the frictional coefficient of the Newtonian case when the Schmidt number approaches infinity. If the coating is a pseudoplastic material, the

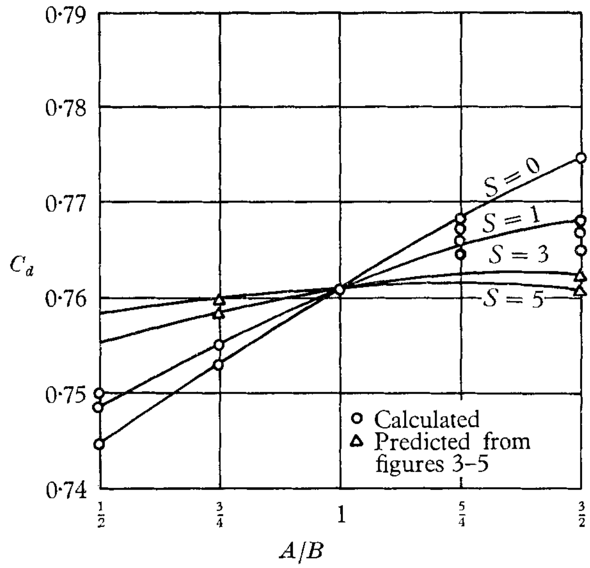

FIGURE 6. Relationship between the frictional coefficient C_d and the parameter A for the Schmidt number $S = 0$, 1, 3 and 5 when $B = 0{\cdot}3$ and $R = 0{\cdot}00$.

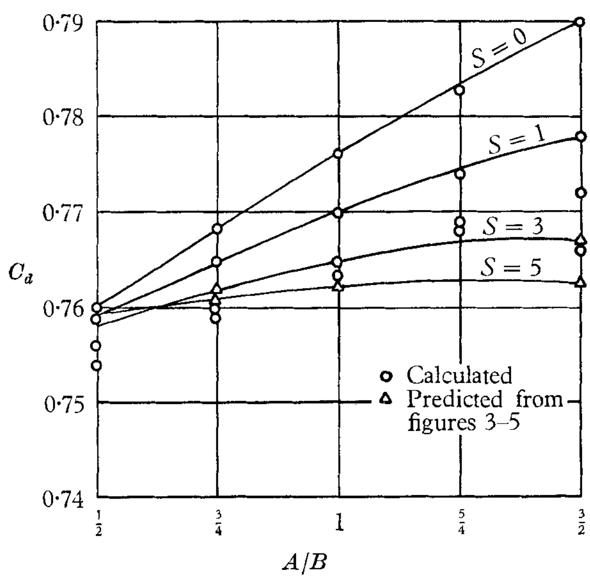

FIGURE 7. Relationship between the frictional coefficient C_d and the parameter A for $S = 0$, 1, 3 and 5 when $B = 0{\cdot}3$ and $R = 0{\cdot}04$.

frictional coefficient in general will increase and approach that of the Newtonian case as the Schmidt number increases; however, for some of the highly pseudo-plastic coatings the frictional coefficient will decrease first and then increase to approach the Newtonian limit as the Schmidt number increases.

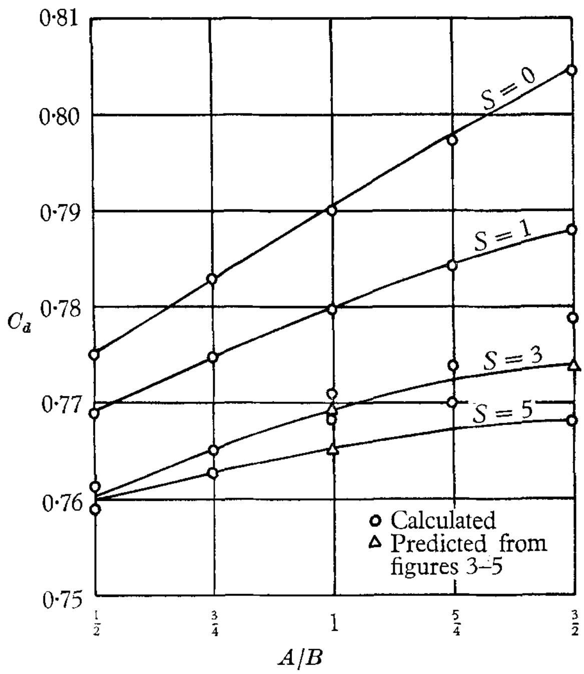

FIGURE 8. Relationship between the frictional coefficient C_d and the parameter A for $S = 0$, 1, 3 and 5 when $B = 0{\cdot}3$ and $R = 0{\cdot}08$.

Case III. *Viscoelastic liquids flowing past the coated wedge*

The case where $g(0) \neq g(\infty) \neq 0$ has not been studied in detail because the results for this case can be predicted qualitatively from the results obtained in case I and case II.

If the external flow now considered is a solution of the coating with the dimensionless concentration $g(\infty)$, it is expected that the frictional coefficient will increase or decrease from that of the homogeneous viscoelastic flow with concentration c_0 as the Schmidt number increases, and will approach a limit as the Schmidt number becomes very large. However, the limit now is the frictional coefficient of the viscoelastic liquid with $g(\infty)$ flowing past the non-coated wedge as can be shown by an analysis similar to that of the previous case. This limit can be obtained from figure 2 and (58) by a suitable choice of the values of R, A and B because

$$R \equiv \gamma c_0, \quad A \equiv \alpha c_0^2(\rho E^3/\eta_0), \quad B \equiv \beta c_0^2(\rho E^3/\eta_0).$$

Since the values for C_d for $S = 0$ and $S \to \infty$ can be obtained from figure 2 and (58), the relationship between C_d and S can then be predicted at least qualitatively for the case considered.

Now, figure 2 and (58) indicate that for dilatant fluids if $g(\infty) < 1$, the skin friction will decrease with increasing Schmidt number; on the other hand, if

$g(\infty) > 1$, C_d increases with increasing Schmidt number. For pseudoplastic fluids, depending on the values of R, A and B, the frictional coefficient C_d may either decrease or increase with increasing Schmidt number when $g(\infty) \gtrless 1$.

The velocity distribution of the flow was obtained for case I and case II from (54) and (52) by calculating η and $f_\eta(\eta)$ for a given value of τ. Similarly, the concentration distribution of the viscoelastic material can be obtained from (43) and (44). Figure 9 shows the velocity profile of the homogeneous viscoelastic flow when $B = 0{\cdot}3$ and $R = 0{\cdot}00$. It indicates that the general form of the velocity distribution of the viscoelastic flows is very similar to that of the Newtonian flow ($A/B = 1$). For the viscoelastic liquids having $0 \leqslant A/B \leqslant 2$, the velocity

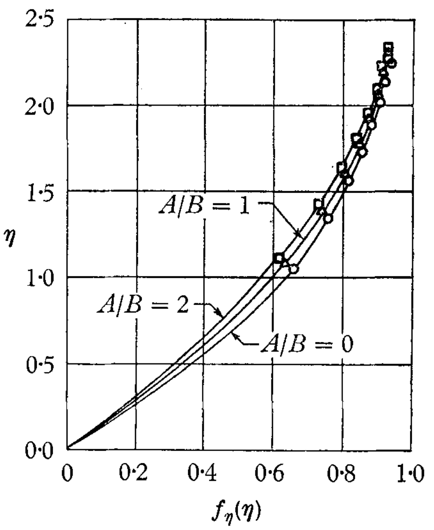

FIGURE 9.

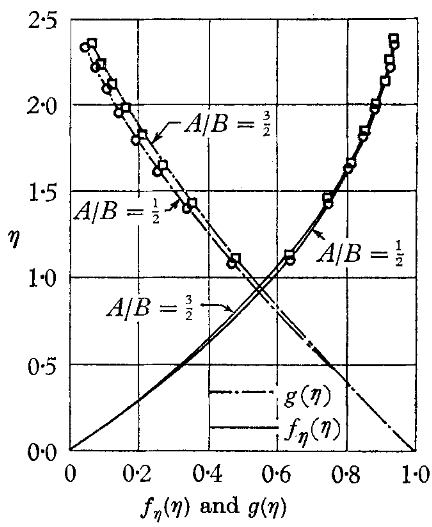

FIGURE 10.

FIGURE 9. Velocity distribution of the homogeneous viscoelastic flow with $B = 0{\cdot}3$ and $R = 0{\cdot}00$.

FIGURE 10. Velocity profile and concentration distribution of the non-homogeneous viscoelastic flow with $S = 1$, $B = 0{\cdot}3$ and $R = 0{\cdot}08$.

profiles of the flow will fall between the two curves of $A/B = 0$ and 2 shown in figure 9.

Figure 10 shows the velocity profile and the concentration distribution of the non-homogeneous viscoelastic flow when $B = 0{\cdot}3$, $R = 0{\cdot}08$ and the Schmidt number $S = 1$. It is seen that the velocity profiles of the liquids with $A/B = \frac{1}{2}$ and $\frac{3}{2}$ are both very similar to each other and the deviation from that of the Newtonian flow is small. The concentration distributions shown in figure 10 indicate that the thickness of the diffusion layers is almost independent of the material properties and has the same order of magnitude as that of the velocity profiles when the Schmidt number equals one. The thickness of the diffusion layer will of course decrease as the Schmidt number increases, but the comparison of figures 9 and 10 shows that the order of magnitude of the boundary-layer thickness is relatively insensitive to the Schmidt number.

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